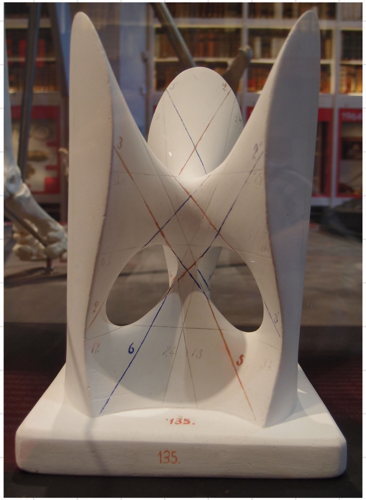
Bott's residue formula in $\mathbb{A}^1$-enumerative geometry

($\mathbb{A}^1$-) enumerative geometry

Example: How many lines are there on a smooth cubic surface?

Clebsch cubic surface

$$x_0^3+x_1^3+x_2^3+x_3^3=(x_0+x_1+x_2+x_3)^3$$



Fermat cubic

$$x_0^3+x_1^3+x_2^3+x_3^3=0$$

$$(u:-\xi u:v:-\xi' v)$$

$$3\cdot 3\cdot 3=27$$

Over $\mathbb{C}$: 27

(Cayley-Salmon 1849)

⇝ enumerative geometry

Over $\mathbb{R}$:

→ 3, 7, 15, 27 (Schläfli 1863)

→ signed count = 3 (Segre 1942)

⇝ real enumerative geometry

in $GW(k)$: $15\langle 1\rangle+12\langle -1\rangle$

(Kass-Wickelgren 2017)

⇝ $\mathbb{A}^1$-enumerative geometry

Grothendieck-Witt ring of k char $k \neq 2$

$GW(k) :=$ group completion of isometry classes of non-degenerate quadratic forms over k

Addition: $\oplus$ $q_1 : V_1 \to k$, $q_2 : V_2 \to k$ $q_1 \oplus q_2 : V_1 \oplus V_2 \to k$

Multiplication: $\otimes$

generators: $q = a_1 x_1^2 + a_2 x_2^2 + \dots + a_n x_n^2$

$\langle a \rangle = [ax^2]$ $a \in k^\times/(k^\times)^2$

relations:
1) $\langle a \rangle \langle b \rangle = \langle ab \rangle$
2) $\langle a \rangle + \langle b \rangle = \langle a+b \rangle + \langle ab(a+b) \rangle$
3) $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle =: h$ hyperbolic form

Witt ring $W(k) := \dfrac{GW(k)}{\mathbb{Z} \cdot h}$

Examples:

1) $GW(\mathbb{C}) \cong \mathbb{Z}$
   $W(\mathbb{C}) \cong \mathbb{Z}/2$

2) $GW(\mathbb{R}) \cong \mathbb{Z} \times \mathbb{Z}$ $\cong$ as a group
   $W(\mathbb{R}) \cong \mathbb{Z}$

How to count lines:

$V := \mathrm{Sym}^3 S^\vee$ $\qquad$ $\mathrm{Sym}^3 S^\vee_{[\ell]}$ = degree 3 poly's on ℓ

$\downarrow$

$G(1,3)$ $\leftarrow$ Grassmannian of lines in $\mathbb{P}^3$

$\quad \ni [\ell]$

$X = \{f = 0\} \subseteq \mathbb{P}^3$ cubic surface $\rightsquigarrow$ $\sigma_f : G(1,3) \to \mathrm{Sym}^3 S^\vee$ section

deg 3

$\sigma_f([\ell]) = f|_\ell$

$\# \text{lines on } X \xleftrightarrow{1:1} \# \text{zeros of } \sigma_f \overset{=}{=} \deg e(V)$

Euler class

In $GW(k)$: Want to compute degree of

$e^{\mathbb{A}^1}(V) \in H^*(G(1,3), K^{MW}_*(\det^{-1} V))$

$= \widetilde{CH}^*(G(1,3), \det^{-1} V)$

oriented chow groups

$\widetilde{CH}^0(\mathrm{Spec}\, k) = GW(k)$

Computing deg e(V) using Bott's residue formula

Equivariant cohomology

$G \curvearrowright X$
- G algebraic group scheme
- X algebraic variety

$$EG \longrightarrow BG = EG/G$$

contractible with free G-action

classifying space

$$X_G := \frac{EG \times X}{(e\cdot g, x) \sim (e, g\cdot x)}$$

$$H^*_G(X) := H^*(X_G)$$

Example:

- $G = \mathbb{C}^*$

  $H^*_G(pt) = H^*(BG) = H^*(\mathbb{C}P^\infty) = \mathbb{Z}[t]$

- $G = (\mathbb{C}^*)^s$

  $H^*_G(pt) = \mathbb{Z}[t_1, \dots, t_s]$

For $V \to X$ G-vector bundle

$\rightsquigarrow$ vector bundle $V_G \longrightarrow X_G$ $\quad e_G(V) := e(V_G)$

Theorem (Bott's residue formula) $\quad G = (\mathbb{C}^*)^s$

$G \curvearrowright X$ with finitely many fixed pts $\quad p_1, \dots, p_n \,,\ \alpha \in H^*_G(X)$

$$\boxed{\deg \alpha = \sum_{j=1}^{n} \deg \frac{i_{p_j}^* \alpha}{e_G(T_{p_j} X)}}$$

$$i_{p_j} : \{p_j\} \to X$$

$$\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[d, "\pi_X"'] & X_G \arrow[d, "\pi_{X_G}"] \\
pt \arrow[r] & BG
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
e(V) \in H^*(X) \arrow[d, "\pi_{X_*}"'] & H^*_G(X) \ni e_G(V) \arrow[l] \arrow[d] \\
\deg e(V) \in H^*(pt) & H^*(BG) \ni \deg e_G(V) \arrow[l, "ev_0"'] \arrow[u, phantom, "\cong \mathbb{Z}[t_1, \dots, t_s]"]
\end{tikzcd}$$

So $\quad \deg e(V) = ev_0 \sum_{\text{fixed pts}} \deg \dfrac{e_G(V_p)}{e_G(T_p X)}$

Back to counting lines: $H_G^\bullet(pt) = \mathbb{Z}[t_0, t_1, t_2, t_3]$

$G = (\mathbb{C}^*)^4 \curvearrowright \mathbb{CP}^3 \rightsquigarrow (\mathbb{C}^*)^4 \curvearrowright G(1,3)$

6 fixed pts: $l_{ij} = \{x_i = x_j = 0\}$

$V = \mathrm{Sym}^3 S^\vee \to G(1,3)$

$$\boxed{\deg e(V) = ev_0 \sum_{\substack{\text{fixed}\\ \text{pts } p}} \deg \frac{e_G(V_p)}{e_G(T_p G(1,3))}}$$

Bott's residue formula

$l_{23} = \{x_2 = x_3 = 0\}$ $\quad \mathrm{Sym}^3 S^\vee_{[l_{23}]}$ has basis $x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3$

G-repr $\quad$ 4 1-dim irred subspaces

$e_G(\mathrm{Sym}^3 S^\vee_{[l_{23}]}) = 3 \cdot t_0 \cdot (2t_0 + t_1)(t_0 + 2t_1) \cdot 3t_1$

$e_G(T_{[l_{23}]} G(1,3)) = (t_0 - t_2)(t_0 - t_3)(t_1 - t_2)(t_1 - t_3)$

Exercise: $\deg e(V) = 27$

Cohomology with Witt valued coefficients

- $\mathcal{W}$ sheaf on Sm_k ← smooth varieties over k
- can take cohomology with coefficients in this sheaf

  eg $H^*(\operatorname{Spec} k, \mathcal{W}) = W(k)$
- have pullback
- proper pushforward with twists

$$H^*(X, \mathcal{W}(\omega_{X/k})) \to H^*(Y, \mathcal{W}(\omega_{Y/k}))$$

- have equivariant cohomology

Theorem (Ananyevskiy):

$$\boxed{H^*(BSL_2, \mathcal{W}) = W(k)[e]}$$

Recall

$H^*(B\mathbb{C}^*) = \mathbb{Z}[t]$

but

$H^*(B\mathbb{C}^*, \mathcal{W}) = W(k)$

Bott's residue formula for cohomology with Witt coeff

<u>Thm</u> (M. Levine)

N = normalizer of $\mathbb{G}_m \overset{t}{\nearrow}$ in $SL_2 \overset{\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}}{\nearrow}$

$N \curvearrowright X$ with finitely many fixed pts

Then for $\alpha \in H^*_N(X, \mathcal{W}(\omega_{X/k}))$

$$\boxed{\deg \alpha = \sum_{\text{fixed pts}} \deg \frac{i_p^* \alpha}{e_N(T_p X)}}$$

$i_p : \{p\} \hookrightarrow X$

Rmk (Levine):

- N is generated by $t = \begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}$ & $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$
- $H^*(BN, \mathcal{W})[e^{-1}] \overset{\sim}{=} H^*(BSL_2, \mathcal{W})[e^{-1}]$

e = Euler class of fundamental repr $SL_2 \curvearrowright \mathbb{A}^2$

Irreducible N-representations (Levine):

In "Motivic Euler characteristics and Witt valued characteristic classes"

$\rho_0^- : N \to GL_1 \qquad t \mapsto 1 \qquad \sigma \mapsto -1$

$\rho_m^- : N \to GL_2 \qquad t \mapsto t^m = \begin{pmatrix} t^m & 0 \\ 0 & t^{-m} \end{pmatrix}, \quad \sigma \mapsto -\begin{pmatrix} 0 & 1 \\ (-1)^m & 0 \end{pmatrix}, \qquad m \in \mathbb{N}$

$\rightsquigarrow$ bundle $\quad \tilde{\mathcal{O}}^{\pm}(m) \to BN$

$$\varepsilon(m) = \begin{cases} +1 & m \equiv 1 \ (4) \\ -1 & m \equiv -1 \ (4) \end{cases}$$

Thm (Levine)

- $$e(\tilde{\mathcal{O}}(m)) = \begin{cases} \varepsilon(m) \cdot m \cdot e & \in H^2(BN, W) & m \text{ odd} \\ \frac{m}{2} \cdot \tilde{e} & \in H^2(BN, W(\gamma)) & m \equiv 2 \ (4) \\ -\frac{m}{2} \cdot \tilde{e} & \in H^2(BN, W(\gamma)) & m \equiv 0 \ (4) \end{cases}$$
  $m \geq 1$

$\tilde{e}^2 = 4e^2$

non trivial elmt of $\mathrm{Pic}(BN)$

- $e(\tilde{\mathcal{O}}^{\pm}(0)) = 0$
- $e(\tilde{\mathcal{O}}^{-}(m)) = -e(\tilde{\mathcal{O}}(m))$

Back to counting lines: $V = \mathrm{Sym}^3 S^\vee \to G(1,3)$

$N \curvearrowright \mathbb{A}^4_k \rightsquigarrow N \curvearrowright G(1,3)$ with 2 fixed pts

with weights w_1 & w_2 $\quad \ell_{01} = \{x_0 = x_1 = 0\}$ & $\ell_{23} = \{x_2 = x_3 = 0\}$

$$V_{[\ell_{23}]} = \underset{\rho_3}{\langle x_0^3, x_1^3 \rangle} \oplus \underset{\rho_1^-}{\langle x_0^2 x_1, x_0 x_1^2 \rangle}$$

$$e^{\mathcal{W}}_N(V_{[\ell_{23}]}) = 3 \cdot w_1^2 \cdot e^2$$

$$e^{\mathcal{W}}_N(T_{[\ell_{23}]} G(1,3)) = (w_1^2 - w_2^2) e^2$$

$$\sum_{\substack{\text{fixed}\\ \text{pts } p}} \deg \frac{e(V|_p)}{e(T_p G(1,3))} = \frac{3 w_1^2 e^2}{(w_1^2 - w_2^2) e^2} + \frac{3 w_2^2 e^2}{(w_2^2 - w_1^2) \cdot e^2} = 3 \in W(k)$$

In $GW(k)$: $\frac{27-3}{2} \cdot h + 3\langle 1 \rangle \in GW(k)$

Conjecture (Clemens 1984):
A general quintic hypersurface in $\mathbb{P}^4$ contains only finitely many smooth rational curves of degree d.

true for $d \leq 11$, unknown else

How many?

$d=1$ (lines): 2875

$d=2$ (conics): 609 250 Katz 1986

$d=3$ (twisted cubics): 317 206 375 Ellingsrud - Strømme 1994

can be computed with Bott's residue formula

formula for general d by Candelas - de la Ossa - Green (1991)

proved by Givental (1996), Lian - Liu - Yau (1997)

Ellingsrud-Strømme: vector bundle $\mathcal{E}_5 \to H_4$ ← twisted cubic in $\mathbb{P}^4$

$$\# \text{ twisted cubics } = \deg(e(\mathcal{E}_5))$$

Levine-P.: $\mathcal{E}_5 \to H_4$ is relatively orientable

and $\deg(e^{\mathcal{W}}(\mathcal{E}_5)) = 765\langle 1\rangle \in W(k)$

In $GW(k)$:

$$\frac{317\,206\,375 - 765}{2} \cdot h + 765\langle 1\rangle \in GW(k)$$

$$e^{\mathcal{W}}(V) \in H^*(X, \mathcal{W}(\det^{-1} V)) \cong H^*(X, \mathcal{W}(\omega_{X/k}))$$

if $\det^{-1} V \otimes \omega_{X/k} \cong \mathcal{L}^{\otimes 2}$

More counts:

n	degree(s)	signature	rank
4	(5)	765	317206375
5	(3,3)	90	6424326
10	(13)	768328170191602020	794950563369917462703511361114326425387076
11	(3,11)	4407109540744680	31190844968321382445502880736987040916
11	(5,9)	313563865853700	1634858783493329027386903535388009 00
11	(7,7)	136498002303600	31226586782010349970656128100205356
12	(3,3,9)	43033957366680	3550223653760462519107147253925204
12	(3,5,7)	5860412510400	67944157218032107464152121768900
12	(5,5,5)	1833366298500	68075954259605149177418598125 00

THANK

YOU!